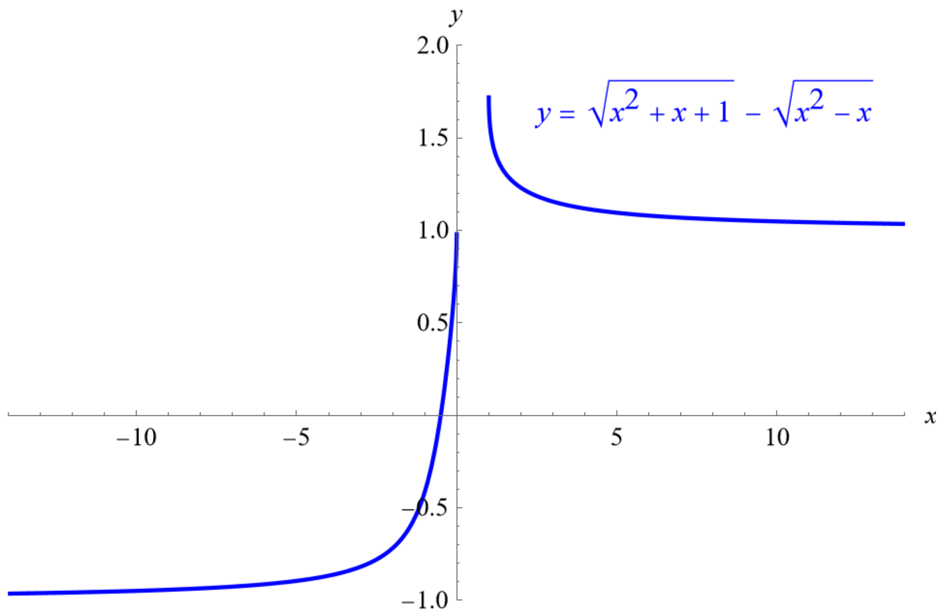
Exercise 22

Use graphs to discover the asymptotes of the curve. Then prove what you have discovered.

$$y = \sqrt{x^2 + x + 1} - \sqrt{x^2 - x}$$

Solution

Below is a graph of the function versus x.



Start by rewriting the difference as a quotient.

$$\begin{aligned}
y &= \sqrt{x^2 + x + 1} - \sqrt{x^2 - x} \\
&= \left(\sqrt{x^2 + x + 1} - \sqrt{x^2 - x}\right) \times \frac{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}} \\
&= \frac{\left(\sqrt{x^2 + x + 1} - \sqrt{x^2 - x}\right)\left(\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}\right)}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}} \\
&= \frac{(x^2 + x + 1) - (x^2 - x)}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}} \\
&= \frac{2x + 1}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}}
\end{aligned}$$

To determine the vertical asymptote(s), set what's in the denominator equal to zero and solve the equation for x.

$$\sqrt{x^2 + x + 1} + \sqrt{x^2 - x} = 0$$

No value of x satisfies this equation, so there are no vertical asymptotes. To determine the horizontal asymptote(s), find the limit of the function as $x \to \pm\infty$. In the limit as $x \to -\infty$, make the substitution, $u = -x$, so that as $x \to -\infty$, $u \to \infty$.

$$\begin{aligned}
\lim_{x\to-\infty} y &= \lim_{x\to-\infty} \frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} \\
&= \lim_{u\to\infty} \frac{2(-u)+1}{\sqrt{(-u)^2+(-u)+1}+\sqrt{(-u)^2-(-u)}} \\
&= \lim_{u\to\infty} \frac{-2u+1}{\sqrt{u^2-u+1}+\sqrt{u^2+u}} \\
&= \lim_{u\to\infty} \frac{-2u+1}{\sqrt{u^2\left(1-\frac{1}{u}+\frac{1}{u^2}\right)}+\sqrt{u^2\left(1+\frac{1}{u}\right)}} \\
&= \lim_{u\to\infty} \frac{-2u+1}{u\sqrt{1-\frac{1}{u}+\frac{1}{u^2}}+u\sqrt{1+\frac{1}{u}}} \\
&= \lim_{u\to\infty} \frac{-2+\frac{1}{u}}{\sqrt{1-\frac{1}{u}+\frac{1}{u^2}}+\sqrt{1+\frac{1}{u}}} \\
&= \frac{-2+0}{\sqrt{1-0+0}+\sqrt{1+0}} \\
&= -1
\end{aligned}$$

Therefore, one horizontal asymptote is $y = -1$. Now find the limit as $x \to \infty$.

$$\begin{aligned}
\lim_{x\to\infty} y &= \lim_{x\to\infty} \frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} \\
&= \lim_{x\to\infty} \frac{2x+1}{x\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+x\sqrt{1-\frac{1}{x}}} \\
&= \lim_{x\to\infty} \frac{2+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x}}} \\
&= \frac{2+0}{\sqrt{1+0+0}+\sqrt{1-0}} \\
&= 1
\end{aligned}$$

Therefore, the other horizontal asymptote is $y = 1$.